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STRING THEORY AND INTEGRABLE SYSTEMS

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Abstract

In this brief review we try to outline the basic structures underlying *integrable* quantum field theory models with *infinite-dimensional* symmetry groups which display *quantum group* symmetries. Certain aspects are treated in some detail: integrable systems of Kadomtsev-Petviashvili type and their reductions appearing in matrix models of strings; Hamiltonian approach to Lie-Poisson symmetries; quantum field theory approach to two-dimensional relativistic integrable models with dynamically broken conformal invariance. All field-theoretic models in question are of relevance to diverse branches of physics ranging from nonlinear hydrodynamics to string theory of fundamental particle interactions at ultra-high energies.

1 Introduction

A current view in theoretical physics is that fundamental laws of Nature can be understood in terms of field-theoretic models in a lower dimensional space-time possessing *infinite-dimensional symmetry groups* and, thus, as a rule being *integrable*. These models are a part of the rich and rapidly developing branch of *string theory* [1]. It is believed that string theory is the most viable candidate for a unified theory of all fundamental interactions at ultra-short distances which, in particular, will unify General Relativity and Quantum Mechanics – one of the major challenges of this century's Physics.

We have in mind two large classes of *integrable models*: conformal field theory (CFT) [2, 3] and massive completely integrable models [4, 5, 6] in $D = 2$ space-time dimensions. Typical examples of CFT are the *rational* CFT's, such as the extensively-studied Wess-Zumino-Novikov-Witten (WZNW) models for various Lie groups G, and models obtained from them by the gauging of different subgroups H of G [7]. In the class of massive integrable models, the Sine-Gordon, (nonabelian) massive Thirring models, Toda

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models for various groups, and the Korteveg-de Vries (KdV)and Kadomtsev-Petviashvili (KP) integrable *soliton* evolution equations and their hierarchies have been thoroughly investigated. The *integrability* which is common to both classes stems from the infinite-dimensional Lie-algebraic structure they share: in the first class this being the Noether symmetry algebra, and in the second, the Hamiltonian structure. The underlying infinite-dimensional symmetries (which manifest themselves through the Virasoro (conformal) [2] and affine Kac-Moody algebras [8] and their generalizations such as the W-algebras [9]) play a crucial role in integrability.

The interrelation between CFT and completely integrable models became clear recently through the appearance of KdV and KP integrable hierarchies in the *matrix model* description [10] of (sub)critical strings (*i.e.*, 2-dimensional gravity interacting with conformal "matter" fields). Thus, it is precisely the integrable field theories which provide the proper framework for incorporation of the huge symmetries of string theory models.

The property of *integrability* is studied both in the classical and the quantum cases. The principal problems which one should investigate are, for the classical case: (a) Classification; (b) variables of the action-angle type; and (c) exact integration of the equations of motion; in the quantum case: (a) Classification; (b) exact scattering amplitudes, and (c) exact correlation functions of local fields off the mass shell. The concepts and tools appropriate to these are found in the theory of infinite-dimensional Lie algebras and groups, and *symplectic geometry*, or equivalently, *Hamiltonian mechanics*. The aim here is to uncover the common *geometrical* foundation of the field theories. In the realm of string theory and the theory of completely integrable systems, the intertwining of Hamiltonian and Lie-group structures in field-theoretic models gives rise to new features, which are clearly seen in the main methods for solving the quantization problem in integrable models: *the quantum inverse scattering method* [11, 12], representation theory of infinite-dimensional Lie algebras [8], and *Quantum Groups* (non-commutative and non-cocommutative Hopf algebras) [13, 12].

The generic integrable models are massive field theories which can be regarded as integrable perturbations of conformal field theories [14]. The latter describe the renormalization group fixed points. Such models have the advantage of being relativistically invariant and classifiable by the conformal models of which they are perturbations. Their essential feature, which shows explicitly the connection with conformal models, is the existence of *multi-Hamiltonian* structures, *i.e.*, the existence of at least a second Hamiltonian structure which is compatible with the canonical R-matrix Kirillov-Kostant structure [15]. The fundamental Poisson brackets corresponding to these Hamiltonian structures (linear R-matrix brackets and quadratic (Sklyanin) R-matrix brackets) arise naturally and are well understood within the classical inverse scattering method [5]. They can also be deduced in the semi-classical limit from the basic algebraic structures:

1. Yang-Baxter equation for the quantum version of the R-matrix;

2. fundamental commutation relations for the quantum transfer matrix, involving the quantum R-matrix as "structure constants",

of the quantum inverse scattering method [11] – the first systematic method for quantization of completely integrable models.

In a related development, Drinfeld [16, 13] was the first to understand the deeper algebraic and geometric nature of classical and quantum completely integrable models. He showed that the algebraic structures 1) and 2) mentioned above were the basic structural relations of *the non-commutative and non-cocommutative Hopf algebras* which were ultimately named *Quantum Groups*. Furthermore, it was realized that a quantum group is a *deformation* of a classical Lie group much in the same way quantum mechanics is a deformation of classical Hamiltonian (symplectic) mechanics [17]. In the semiclassical limit the basic quantum group algebraic structures 1) and 2) go over into a special Hamiltonian structure on the classical Lie group G, called the *Lie-Poisson structure*, which is compatible with the group multiplication. This is precisely the class of Hamiltonian structures given by the quadratic fundamental R-matrix Poisson brackets mentioned above in the context of classical completely integrable models.

The concept of *quantum group symmetries* in integrable quantum field and statistical mechanics models has led to several developments: from generalization (*i.e.*, *q-deformation*) of the internal and space-time symmetries in quantum field theory and the connection between spin and statistics, to quantum magnetic chains and the dynamics of critical phenomena [18]. Furthermore, the concepts of integrability and perturbations around exactly solvable theories have found their place in models of elementary particles such as in quantum chromodynamics [19].

In what follows, some of the topics mentioned above will be discussed in greater detail.

2 Matrix Models of Non-Perturbative Strings and Integrability

2.1 Conventional Perturbative String Theory

The standard geometrical formulation of *perturbative* string theory [20] provides the following prescription for calculating physical observables (fermionic degrees of freedom are discarded for simplicity): to construct scattering amplitudes one considers functional integrals over (Euclidean) string worldsheets $\Sigma_{A,G}$ – smooth Riemann surfaces embedded in D-dimensional (Euclidean) space-time R^D of genus G and area A:

$$Z_{string} = \sum_{G=0}^{\infty} g^{2G} \int dA\, e^{-\Lambda A} Z_{A,G} \tag{1}$$

$$Z_{A,G} = \int [\mathcal{D}h]\, \mathcal{D}X\, \exp\{-S_{string}[X,h]\} \prod_{i=1}^{n} \int d^2\sigma_i \sqrt{h}\, V(X,h;k_i) \tag{2}$$

$$\begin{aligned} S_{string} = \frac{1}{2}\int d^2\sigma \Big[\sqrt{h}\left(h^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + \Phi(X) R^{(2)}(h)\right) \\ + \epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X)\Big] \end{aligned} \tag{3}$$

Here the following notations are used: g denotes the string coupling constant, Λ is the cosmological constant, the string action S_{string} represents a typical $D = 2$ conformally invariant field theory describing $D = 2$ gravity given by the world-sheet metric $h_{ab}(\sigma)$ and coupled to world-sheet matter fields $X^\mu(\sigma)$ describing the embedding of $\Sigma_{A,G}$ in R^D. The functionals $G_{\mu\nu}, \Phi, B_{\mu\nu}$ represent the space-time dilaton-gravity multiplet. String interactions are given in (1),(2) by geometrical splitting and joining of individual string world-sheets thus creating handles on the total world-sheet, whereas the asymptotic incoming and outgoing string states are given by vertex operators $V(X,h;k)$ (k denoting the momentum of the asymptotic state).

Henceforth, for simplicity, the vertex operators V's in (2) will be suppressed, *i.e.*, we shall concentrate on the string partition function.

The most difficult part of calculating (2) is handling the functional measure $[\mathcal{D}h]$ over the space of all *gauge-inequivalent* classes of metrics h_{ab} on $\Sigma_{A,G}$ w.r.t. reparametrization and Weyl conformal invariance. In the conformal gauge, $h_{ab} = e^\phi \hat{h}_{ab}(\tau)$ where ϕ is the Weyl conformal factor and $\hat{h}_{ab}(\tau)$ is a reference metric with constant curvature $R^{(2)}\left(\hat{h}_{ab}\right)$ and which depends in general on the moduli $\{\tau\}$ of the Riemann surface $\Sigma_{A,G}$, the standard Faddeev-Popov gauge-fixing procedure yields [20, 21]:

$$[\mathcal{D}h] = \delta\left(h_{ab} - e^\phi \hat{h}_{ab}(\tau)\right) \Delta_{\Phi\Pi} \mathcal{D}\phi\left(d\tau\right) \tag{4}$$

with the Faddeev-Popov determinant $\Delta_{\Phi\Pi}$ giving rise to the well-known conformal anomaly.

An important result about the entropy of random surfaces with fixed area A, first obtained by Zamolodchikov [22] in the semi-classical approximation and subsequently strengthened in [23], states that for large A:

$$Z_{A,G} \simeq_{A\to\infty} const_G e^{\Lambda_c A} A^{-\chi(1-\gamma_0/2)-1} \tag{5}$$

where Λ_c denotes a critical value of the cosmological constant Λ, $\chi = 2(1-G)$ is the Euler characteristic of the surfaces, and γ_0 denotes a critical exponent depending on the world-sheet matter fields. Relation (5) implies for the string partition function (1):

$$Z_{string} \simeq (\Lambda - \Lambda_c)^{2-\gamma_0} \sum_{G=0}^{\infty} const_G \left(\frac{g^2}{(\Lambda - \Lambda_c)^{2-\gamma_0}} \right)^G \tag{6}$$

which shows that one can obtain the complete nonperturbative result for Z_{string} by taking the *double scaling* limit:

$$\Lambda \longrightarrow \Lambda_c \ , \quad g^2 \longrightarrow 0 \quad \textit{such that} \quad g^2_{ren} \equiv \left(\frac{g^2}{(\Lambda - \Lambda_c)^{2-\gamma_0}} \right) = fixed \tag{7}$$

2.2 Lattice Regularization of String Theory; Matrix Model Formulation

It has been found [24] that statistical mechanical models of random matrices ("matrix models" for short) provide an adequate apparatus for the nonperturbative description of lattice-regularized string theory based on the method of *random triangulation* (and, more generally, random polygonization) of the (Euclidean) string world-sheet. Furthermore, ways have been proposed in refs. [10] for the correct implementation of the continuum limit as a double scaling limit (7), which admits exact solutions in string theory.

Let us note that, whereas matrix model formulation of random surfaces is adequate for solving integrable lattice models of planar statistical mechanics, its application to genuine string theory is limited so far to the case of $D \leq 2$ dimensional embedding space. Nonetheless, the exact solvability of matrix models provides an important testing ground and qualitative hints for nonpertubative string theory solutions in realistic cases (for extensive reviews, see [26]).

Since our main aim is to clarify the emergence of integrability structures in matrix models of string theory, we shall consider for illustrative purpose the simplest one-matrix model with partition function given by:

$$Z = \int d^{N^2} M e^{-NV(M)} \quad , \quad V(M) = \sum_{k \geq 0} t_k \left(\frac{N}{\beta} \right)^{k/2-1} Tr\, M^k \tag{8}$$

with $M = \|M_{ij}\|$ being a $N \times N$ hermitian matrix. In ordinary perturbation theory defined in terms of a free propagator $\langle M_{ij} M_{kl} \rangle_0 \sim N^{-1} \delta_{ik} \delta_{jl}$ and k-leg vertices with weights $t_k N \left(\frac{N}{\beta} \right)^{k/2-1}$, each diagram Γ gives a contribution of the form:

$$\prod_{k \geq 3} \left(t_k N \left(\frac{N}{\beta} \right)^{k/2-1} \right)^{V_k(\Gamma)} N^{(L(\Gamma) - P(\Gamma))} = \mathcal{W}_\Gamma \left[\{t\} \right] N^{\chi(\Gamma)} \left(\frac{\beta}{N} \right)^{-L(\Gamma)} \tag{9}$$

On the l.h.s. of (9) $V_k(\Gamma)\,,P(\Gamma)$ and $L(\Gamma)$ denote the numbers of k-leg vertices, propagators (links) and closed loops (faces) of Γ, whereas on the r.h.s. $\chi(\Gamma) = L(\Gamma) - P(\Gamma) + \sum_{k\geq 3} V_k(\Gamma)$ denotes the Euler characteristic of the two-dimensional polygonized surface spanned by Γ, and $\mathcal{W}_\Gamma\left[\{t\}\right]$ the product of the vertex weights. Clearly, $L(\Gamma) \equiv A(\Gamma)$ can be understood as the area of Γ. Thus the partition function (8) can be written as:

$$Z = \exp\left\{ \sum_{conn.\ surfaces\ \Gamma} N^{\chi(\Gamma)} e^{-(\ln\beta/N)A(\Gamma)+\ln\mathcal{W}_\Gamma[\{t\}]} \right\} \tag{10}$$

i.e., the *free energy* $\ln Z$ of the matrix model (10) represents the discretized regularized partition function of random surfaces (more precisely, pure $D = 2$ gravity with action $A(\Gamma)$ interacting with matter with action $\ln\mathcal{W}_\Gamma$) upon making the following identifications (comparing with (1)-(2), (5)): $1/N \simeq g$ (bare string coupling constant), $N/\beta \simeq e^{-(\Lambda-\Lambda_c)}$ (Λ - the cosmological constant) and the double scaling limit (7) takes the form of a special continuum limit:

$$\frac{N}{\beta} \longrightarrow 1\ ,\ N \longrightarrow \infty\ ,\ such\ that\ g^2_{ren} \equiv \left[\beta^2\left(\frac{\beta}{N} - 1\right)^{2-\gamma_0}\right]^{-1} = fixed \tag{11}$$

The explicit solution for the partition function (8) is obtained by the method of orthogonal polynomials [25]. Diagonalizing the hermitian matrix $M = U diag\,(\lambda_1,\ldots,\lambda_N)\,U^{-1}$ and integrating over angle variables in U, one gets (rescaling $M \longrightarrow (\beta/N)^{\frac{1}{2}}\,M$):

$$Z = \int \prod_{i=1}^N d\lambda_i\,\Delta(\lambda)\exp\left\{-\beta\sum_{i=1}^N V\left(\lambda_i\right)\right\}\Delta(\lambda) \tag{12}$$
$$V(\lambda_i) = \sum_{k\geq 0} t_k\lambda_i^k,\quad \Delta(\lambda) = \prod_{i<j}\left(\lambda_i - \lambda_j\right)$$

Introducing a complete set of orthogonal polynomials $P_n(\lambda) = \lambda^n +$ lower order terms:

$$\int d\lambda P_n(\lambda) e^{-\beta V(\lambda)} P_m(\lambda) = h_n\delta_{nm} \tag{13}$$

and re-expressing the Vandermonde determinant as $\Delta(\lambda) = \det\|P_i(\lambda_j)\|$, the integrals in (12) factorize, and give, using (13):

$$Z\left[\{t\}\right] = \prod_{n=0}^N h_n\left(\{t\}\right) \tag{14}$$

where the dependence on the parameters of the random matrix potential is indicated. Explicit solutions for $h_n\left(\{t\}\right)$ can be found by solving the flow equations w.r.t. t_k which correspond to integrable lattice hierarchies, as briefly discussed in the next subsection.

2.3 Differential Integrable Hierarchies from Matrix Models

The appearance of integrable hierarchies in the continuum (double scaling) limit (11) has been extensively discussed in the literature (see [26] and refs. therein). It is very interesting, however, that the flow equations can be obtained directly from discrete matrix models even before taking the continuum limit [27, 28]. This reveals their close connection to topological field theories [29].

The above result is obtained most easily by the method of orthogonal polynomials (13). On the Hilbert space spanned by $\{P_n(\lambda)\}_{n\geq 0}$, one introduces two conjugate operators $\mathcal{Q}, \mathcal{P}$ with matrix elements defined by:

$$h_n \mathcal{Q}_{mn} = \int d\lambda P_n(\lambda) e^{-\beta V(\lambda)} \lambda P_m(\lambda) \tag{15}$$

$$h_n \mathcal{P}_{mn} = \int d\lambda P_n(\lambda) e^{-\beta V(\lambda)} \frac{d}{d\lambda} P_m(\lambda) \tag{16}$$

From (15), (16) one easily gets the matrix model string equation (the second one below):

$$\mathcal{P} = \beta \left(V'(\mathcal{Q})\right)_{(-)} \longrightarrow \left[\beta \left(V'(\mathcal{Q})\right)_{(-)}, \mathcal{Q}\right] = 1 \tag{17}$$

where the subscript $(-)$ denotes the strictly lower-diagonal part of the corresponding matrix. The string equation yields recurrsion relations for the matrix elements (15) of $\mathcal{Q}$.

It is straightforward to deduce from (15) and (13) the following flow equations:

$$\frac{\partial P_n(\lambda)}{\partial t_r} = \mathcal{Q}^r_{(-)\,nm} P_m(\lambda) \quad , \quad \mathcal{Q}_{nm} P_m(\lambda) = \lambda P_n(\lambda) \tag{18}$$

$$\frac{\partial \mathcal{Q}}{\partial t_r} = \left[\, \mathcal{Q}^r_{(-)}, \mathcal{Q} \right] \tag{19}$$

Eq. (19) is the integrability condition for eqs. (18) and it is compatible with the string equation (17). One can identify (19) with the Lax form of the flow equations of the integrable Toda lattice hierarchy by inserting into (19) the explicit form of the $\mathcal{Q}$ matrix elements:

$$\mathcal{Q}_{n,n+1} = 1 \,, \quad \mathcal{Q}_{n,n} \equiv \mathcal{S}_{n-1} \,, \quad \mathcal{Q}_{n+1,n} = \frac{h_{n+1}}{h_n} \equiv \mathcal{R}_n \equiv e^{\phi_n - \phi_{n-1}} \tag{20}$$

the rest being zero owing to the recurrence relations for orthogonal polynomials. Eqs. (19) are now Hamiltonian, and the lowest one (with $r = 1$) is generated by the Toda lattice Hamiltonian:

$$H_{Toda} = \frac{1}{2}\sum_n \mathcal{S}_n^2 + \sum_n \left(e^{\phi_{n+1}-\phi_n} - 1\right), \quad \{\mathcal{S}_n\,,\,\phi_m\} = \delta_{nm} \tag{21}$$

Following [28], it is possible to replace the discrete lattice integrable hierarchy (19) by a differential hierarchy at each fixed lattice site n where the continuum variable is $x = t_1$. Indeed, from the first ($r = 1$) flow eqs. (18) and (19) yielding

$$\frac{\partial P_{n+1}}{\partial t_1} = \mathcal{R}_n P_n\ , \quad \frac{\partial \mathcal{R}_n}{\partial t_1} = \mathcal{R}_n\left(\mathcal{S}_{n-1} - \mathcal{S}_n\right)\ , \quad \frac{\partial \mathcal{S}_n}{\partial t_1} = \mathcal{R}_n - \mathcal{R}_{n+1} \tag{22}$$

one obtains (up to gauge transformation $P_n \longrightarrow \psi_n = \exp\{\int dt_1'\,\mathcal{S}_{n-1}\}\,P_n = h_n^{-1}P_n$, and similarly $\mathcal{Q}_{nm} \longrightarrow \hat{\mathcal{Q}}_{nm} = h_n^{-1}\mathcal{Q}_{nm}h_m$):

$$\begin{aligned} \lambda\psi_n = \hat{\mathcal{Q}}_{nm}\psi_n &= h_n^{-1}\left(P_{n+1} + \mathcal{S}_{n-1}P_n + \mathcal{R}_{n-1}P_{n-1}\right) \\ &= \left[\partial + \mathcal{R}_n\left(\partial - \mathcal{S}_n\right)^{-1}\right]\psi_n \end{aligned} \tag{23}$$

where $\partial \equiv \frac{\partial}{\partial t_1}$. Once again using (22), one can rewrite the discrete evolution eqs. (18), (19) in a differential Lax form for a fixed lattice site n:

$$\frac{\partial \psi}{\partial t_r} = \left(L^r\right)_+\psi \quad , \quad \frac{\partial L}{\partial t_r} = \left[\left(L^r\right)_+\,,\,L\right] \tag{24}$$

where $r \geq 2$, $\psi \equiv \psi_n(t_1)$ and the subscript $+$ indicates taking the purely differential part of the corresponding pseudo-differential Lax operator (cf. last equality in (23)):

$$L = \partial + A\left(\partial - B\right)^{-1} \quad , \quad A \equiv \mathcal{R}_n(t_1) \quad , \quad B \equiv \mathcal{S}_n(t_1) \tag{25}$$

Eqs. (24), (25) are immediately recognized as the 2-boson reduction of KP integrable hierarchy (see subsections 3.2 and 3.3).

Using generalization of the method of orthogonal polynomials, it is possible to derive flow equations for integrable hierarchies also in the general case of multi-matrix models (describing random surfaces interacting with q different types of "matter") :

$$Z = \int \prod_{i=1}^{q} d^{N^2}M_i\ \exp\left\{-\sum_{i=1}^{q}\left(Tr\,M_iM_{i+1} + \sum_{k\geq 0} t_{i,k}\,Tr\,M_i^k\right)\right\} \tag{26}$$

without passing to the continuum limit [28]. The appropriate generalization of (25) now reads:

$$L_q = \partial + \sum_{l=1}^{q} A_l\left(\partial - B_l\right)^{-1}\left(\partial - B_{l+1}\right)^{-1}\ldots\left(\partial - B_q\right)^{-1} \tag{27}$$

which is a $2q$-boson reduction of KP integrable hierarchy (see subsections 3.2 and 3.3 for more details).

Finally, returning to the string partition function (14), one can show that $Z[\{t\}] = \tau_N(t)$ – the Toda lattice τ-function [27] subject to the so called Virasoro constraints:

$$\mathcal{L}_s Z[\{t\}] = 0 \quad , \; s \geq -1 \; ; \quad [\mathcal{L}_r \, , \, \mathcal{L}_s] = (r-s)\mathcal{L}_{r+s} \tag{28}$$

$$\mathcal{L}_{s\geq 0} = \sum_{k\geq 0} k t_k \frac{\partial}{\partial t_{k+s}} + \sum_{k=0}^{s} \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_{s-k}}$$
$$\mathcal{L}_{s<0} = \sum_{k\geq 0} k t_k \frac{\partial}{\partial t_{k-s}} - \sum_{k=0}^{|s|-1} \frac{\partial}{\partial t_{-k}} \frac{\partial}{\partial t_{s+k}}$$

which are equivalent to the constraints on the pertinent integrable hierarchies imposed by the "string" equation (17). They are in fact Ward identities corresponding to the symmetry $\delta_s M = \varepsilon_s M^{s+1}$, $s \geq -1$ of (9). Similar relations hold for multi-matrix models. In this case the continuum (double scaling) limit $Z[\{t\}]$ can be identified with τ-functions of reduced KP hierarchies subject to the so-called W-constraints (generalizations of (28) which span W-algebras; cf. (51) below).

3 Integrable Systems in Classical Physics: Geometric Formulation

This section is devoted to a brief review of some principal structures and properties of integrable systems in classical mechanics and field theory. We first recall the notion of integrability.

Complete integrability: Consider a Hamiltonian system with n degrees of freedom possessing a standard Hamiltonian structure with Hamiltonian $H(p,q)$ and Poisson bracket $\{\cdot,\cdot\}$. A Hamiltonian system is called *completely* (or Liouville) *integrable* if it has n independent integrals of motion I_k, $k = 1,\ldots,n$, which are in involution: $\{I_i, I_j\} = 0$. For such a system one can find the *action-angle* canonical variables and write explicitly the general solution to the equations of motion.

Lax formulation: For infinite-dimensional (field theory) integrable Hamiltonian systems, there exists a convenient Lax (or "zero-curvature") formulation [5]. In the Lax formulation, the phase space of the Hamiltonian system is parametrized by elements L taking values in some Lie algebra $\mathcal{G}$ and the dynamical equations of motion can be written in terms of a Lax pair L, P, the latter also taking values in $\mathcal{G}$, as the Lax-type equation:

$$\frac{dL}{dt} = [L\,,\,P] \tag{29}$$

The Lax formulation leads straightforwardly to the construction of the involutive integrals of motion. Any Ad-invariant function $I(L)$ on $\mathcal{G}$ is a constant of motion. It can be shown that any completely integrable Hamiltonian system admits a Lax representation (at least locally) [30].

A very wide class of integrable models can be constructed in the approach developed by Adler, Kostant and Symes, and by Reyman and Semenov-Tyan-Shansky (AKS-RS scheme) [31, 15] having roots in the group coadjoint orbit method [32].

3.1 AKS-RS Scheme

Let G denote a Lie group and $\mathcal{G}$ be its Lie algebra. G acts on $\mathcal{G}$ by the adjoint action: $Ad(g)\,X = gXg^{-1}$, with $g \in G$ and $X \in \mathcal{G}$. Let $\mathcal{G}^*$ be the dual space of $\mathcal{G}$ relative to a non-degenerate bilinear form $\langle\cdot|\cdot\rangle$ on $\mathcal{G}^* \times \mathcal{G}$. The corresponding coadjoint action of G on $\mathcal{G}^*$ is obtained from the duality of $\langle\cdot|\cdot\rangle$: $\langle Ad^*(g)U|X\rangle = \langle U|Ad(g^{-1})X\rangle$. The infinitesimal versions of the adjoint and coadjoint transformations (for $g = \exp Y$) will be denoted by $ad(Y)X = [\,Y\,,\,X\,]$ and $\langle ad^*(Y)U|\,X\rangle = -\langle U|\,[\,Y\,,\,X\,]\rangle$, respectively. When $\mathcal{G}$ is endowed with an ad-invariant bilinear form $(\cdot,\cdot)$ (Killing form) allowing to identify $\mathcal{G}^*$ with $\mathcal{G}$, one has $ad^*(Y)U = [\,Y\,,\,U\,]$.

There exists a natural Poisson structure on the space $C^\infty\,(\mathcal{G}^*, \mathbb{R})$ of smooth, real-valued functions on $\mathcal{G}^*$ called Kirillov-Kostant (KK) bracket, given by:

$$\{F\,,\,H\}(U) = -\Big\langle U\Big|\big[\,\nabla F(U)\,,\,\nabla H(U)\,\big]\Big\rangle \tag{30}$$

where $F, H \in C^\infty\,(\mathcal{G}^*, \mathbb{R})$, the gradient $\nabla F\,:\,\mathcal{G}^* \longrightarrow \mathcal{G}$ is defined by the formula $\frac{d}{dt}F\,(U + tV)\,|_{t=0} = \langle V\,|\,\nabla F(U)\rangle$ and $[\cdot\,,\,\cdot]$ is the Lie commutator on $\mathcal{G}$. On each orbit of G in $\mathcal{G}^*$ the Poisson bracket (30) gives rise to a non-degenerate symplectic structure. For any Hamiltonian function H on such an orbit one can write a Hamiltonian equation of motion $dU/dt = ad^*(\nabla H(U))U$ ($= [\,\nabla H(U)\,,\,U\,]$ when $\mathcal{G}$ admits a Killing form).

In ref.[15] the R-operator (generalized R-matrix) was introduced as a linear map from a Lie algebra $\mathcal{G}$ to itself such that the bracket:

$$[X, Y]_R \equiv \frac{1}{2}[RX, Y] + \frac{1}{2}[X, RY] \tag{31}$$

defines a second Lie-bracket structure on $\mathcal{G}$, or equivalently, defines a second Lie algebra $\mathcal{G}_R$ isomorphic to $\mathcal{G}$ as a vector space. The Jacobi identity for the R-commutator (31) implies that the modified Yang-Baxter equation (YBE) for the R-matrix should hold (for arbitrary $X_{1,2,3} \in \mathcal{G}$):

$$(32) \quad \sum_{cyclic(1,2,3)} \Big[X_1 \, , \, \big[RX_2 \, , \, RX_3 \big] - R\Big(\big[RX_2 \, , \, X_3 \big] + \big[X_2 \, , \, RX_3 \big] \Big) \Big] = 0$$

A sufficient condition for the fulfilment of (32) is:

$$(33) \qquad \big[RX \, , \, RY \big] - R\Big(\big[RX \, , \, Y \big] + \big[X \, , \, RY \big] \Big) = -\alpha \big[X \, , \, Y \big]$$

where α is an arbitrary constant. Eq. (33) is usually written in terms of the "ordinary" r-matrix $r \in \mathcal{G} \otimes \mathcal{G}$ which is isomorphic to $R \in \mathcal{G} \otimes \mathcal{G}^*$ via the Killing form [1]:

$$(34) \qquad \Big[\overset{12}{r} , \overset{13}{r} \Big] + \Big[\overset{13}{r} , \overset{23}{r} \Big] + \Big[\overset{12}{r} , \overset{23}{r} \Big] = ad - invariant$$

$$\overset{12}{r} \equiv r_{ij} T^i \otimes T^j \otimes 1 \in \mathcal{U}\,(\mathcal{G}) \otimes \mathcal{U}\,(\mathcal{G}) \otimes \mathcal{U}\,(\mathcal{G}) \quad , \quad etc.$$

where $\mathcal{U}\,(\mathcal{G})$ is the universal enveloping algebra. Classification of all one-parameter solutions to (34) for simple Lie algebras $\mathcal{G}$ is given in refs.[33].

A new KK-type Poisson bracket $\{\cdot, \cdot\}_R$, called R-bracket, can be introduced on $\mathcal{G}^*_R \simeq \mathcal{G}^*$ with the help of (31) by substituting the usual Lie commutator $[\cdot, \cdot]$ for the R-Lie commutator $[\cdot, \cdot]_R$ in (30):

$$(35) \qquad \{F \, , \, H\}_R(U) = -\Big\langle U \Big| \big[\nabla F(U) \, , \, \nabla H(U) \big]_R \Big\rangle$$

A function H on $\mathcal{G}^*$ is called Ad^*-invariant (Casimir) if $H[Ad^*(g)U] = H[U]$. Infinitesimally $ad^*(\nabla H(U))(U) = 0$ for each $U \in \mathcal{G}^*$. Then it can be shown that [15]:

1. all ad^*-invariant functions are in involution with respect to both brackets (30) and (35);
2. the Hamiltonian equation on $\mathcal{G}^* \simeq \mathcal{G}^*_R$ takes the following generalized Lax form:

$$(36) \qquad \frac{dU}{dt} = \frac{1}{2} ad^* \Big(R \left(\nabla H(U) \right) \Big) U \; = \; \Big[\frac{1}{2} R \left(\nabla H(U) \right) , \, U \Big]$$

where the second equality holds when $\mathcal{G}$ admits a Killing form. Eq.(36) can be obtained from a variational principle with the following geometric action:

$$(37) \qquad \mathcal{W}[U] = -\int \Big\langle U \Big| \mathcal{Y}_R(U) \Big\rangle - \int dt \, H[U]$$

$$(38) \qquad dU = ad^*_R \left(\mathcal{Y}_R(U) \right) U \quad \longrightarrow \quad d\mathcal{Y}_R = \frac{1}{2} \left[\mathcal{Y}_R \, , \, \mathcal{Y}_R \right]_R$$

---

[1] $r = r_{ij} T^i \otimes T^j$ and $\frac{1}{2} RX = T^i r_{ij} \left(T^j, X \right)$, where $\left\{ T^i \right\}$ denotes a basis in $\mathcal{G}$.

where the integrals in (37) are along a smooth curve in the phase space $\mathcal{G}^*$; $H[U]$ is a Casimir on $\mathcal{G}^*$; $\mathcal{Y}_R$ is the Maurer-Cartan one-form on $\mathcal{G}_R$. The dependence on $U \in \mathcal{G}^*$ of $\mathcal{Y}_R$ is determined from the first eq.(38) with the R-coadjoint action:

$$
\begin{aligned}
ad^*_R(X)U &= \frac{1}{2}\Big(ad^*(RX)U + R^*(ad^*(X)U)\Big) \\
&= \frac{1}{2}\big[\, RX \,,\, U \,\big] - \frac{1}{2}R\Big(\big[\, X \,,\, U \,\big]
\end{aligned}
\tag{39}
$$

where the second equality in (39) holds when $\mathcal{G}$ admits a Killing form. Hence the above AKS-RS technique leads to a construction of completely integrable systems in which the complete set of integrals of motion in involution coincides with the set of independent Casimir functions on $\mathcal{G}^*$.

A realization of this scheme arises when the Lie algebra $\mathcal{G}$ decomposes as a vector space into two subalgebras $\mathcal{G}_+$ and $\mathcal{G}_-$, *i.e.* $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$. Let $P_\pm$ be the corresponding projections on $\mathcal{G}_\pm$. Then $R = P_+ - P_-$ satisfies the modified YBE (32). In this case eqs.(31), (35), (36) and (39) take the following form:

$$
\Big[\, X \,,\, Y \,\Big]_R = \Big[\, X_+ \,,\, Y_+ \,\Big] - \Big[\, X_- \,,\, Y_- \,\Big] \tag{40}
$$

$$
\begin{aligned}
\Big(ad^*_R(X)U\Big)_\pm &= \mp\Big[\, X_\mp \,,\, U_\pm \,\Big]_\pm \\
\Big\{ \Big\langle U_\mp \Big| X_\pm \Big\rangle , \Big\langle U_\mp \Big| Y_\pm \Big\rangle \Big\} &= \pm \Big\langle U_\mp \Big| \Big[\, X_\pm \,,\, Y_\pm \,\Big] \Big\rangle \\
\frac{dU}{dt} + \left[\left(\frac{\delta H}{\delta U} \right)_+ \,,\, U \right] &= 0
\end{aligned}
\tag{41}
$$

where the following notations have been used:

$$
X_\pm = P_\pm X \in \mathcal{G}_\pm \;\; , \;\; U_\pm = P^*_\mp U \in (\mathcal{G}_\mp)^* \;\; , \;\; [\, X \,,\, U \,]_\pm = P^*_\mp\Big([\, X \,,\, U \,]\Big) \in (\mathcal{G}_\mp)^* \tag{42}
$$

3.2 Algebra of Pseudo-Differential Operators and Integrable Hierarchies of Kadomtsev-Petviashvili type

Here the AKS-RS construction will be illustrated for $\mathcal{G} = \Psi\mathcal{DO}$ - the algebra of pseudo-differential operators on the circle. An arbitrary pseudo-differential operator $X(x, D_x) = \sum_{k \geq -\infty} X_k(x) D_x^k$ is conveniently represented by its symbol [34] which is a Laurent series in the variable ξ: $X(\xi, x) = \sum_{k \geq -\infty} X_k(x)\xi^k$. The operator multiplication corresponds to the following symbol multiplication:

$$X(\xi, x) \circ Y(\xi, x) = \sum_{N \geq 0} \frac{1}{N!} \frac{\partial^N X}{\partial \xi^N} \frac{\partial^N Y}{\partial x^N} \tag{43}$$

determining a Lie algebra structure given by: $[X, Y] \equiv (X \circ Y - Y \circ X)$.

An invariant, non-degenerate bilinear form can be introduced on $\Psi\mathcal{DO}$:

$$\left\langle L | X \right\rangle \equiv Tr_A\,(LX) = \int dx\, Res_\xi \left(L(\xi, x) \circ X(\xi, x) \right) \tag{44}$$

which allows identification of the dual space $\Psi\mathcal{DO}^*$ with $\Psi\mathcal{DO}$.

There exist three natural decompositions of $\mathcal{G} = \Psi\mathcal{DO}$ into a linear sum of two subalgebras $\mathcal{G} = \mathcal{G}_+^\ell \oplus \mathcal{G}_-^\ell$ labelled by the index ℓ taking values $\ell = 0, 1, 2$:

$$\begin{aligned} \mathcal{G}_+^\ell &= \left\{ X_+ \equiv X_{\geq \ell} = \sum_{i=\ell}^{\infty} X_i(x) D^i \right\}, \\ \mathcal{G}_-^\ell &= \left\{ X_- \equiv X_{< \ell} = \sum_{i=-\ell+1}^{\infty} X_{-i}(x) D^{-i} \right\} \end{aligned} \tag{45}$$

Correspondingly the dual spaces to subalgebras $\mathcal{G}_\pm^\ell$ are given by:

$$\begin{aligned} \mathcal{G}_+^{\ell\,*} &= \left\{ L_- \equiv L_{< -\ell} = \sum_{i=\ell+1}^{\infty} D^{-i} \circ u_{-i}(x) \right\}, \\ \mathcal{G}_-^{\ell\,*} &= \left\{ L_+ \equiv L_{\geq -\ell} = \sum_{i=-\ell}^{\infty} D^{i} \circ u_{i}(x) \right\} \end{aligned} \tag{46}$$

Note that in (46) the differential operators are to the left. Henceforth, we shall skip the sign $\circ$ in symbol products for brevity.

After defining $R_\ell = P_+ - P_-$ for each of the three cases, eqs.(40) take the form:

$$\begin{aligned} [X, Y]_{R_\ell} &= [X_{\geq \ell}, Y_{\geq \ell}] - [X_{< \ell}, Y_{< \ell}], \\ ad^*_{R_\ell}(X) L &= [X_{\geq \ell}, L_{< -\ell}]_{< -\ell} - [X_{< \ell}, L_{\geq -\ell}]_{\geq -\ell} \end{aligned} \tag{47}$$

Choosing the infinite set of independent Casimir functions in the form:

$$H_{m+1} = \frac{1}{m+1} \int dx\, Res\, L^{m+1} \quad , \; m = 0, 1, 2, \ldots \tag{48}$$

the three decompositions (46) of $\Psi\mathcal{DO}$ labelled by $\ell = 0, 1, 2$ yield, according to the AKS-RS scheme, three different integrable hierarchies – the standard KP hierarchy ($\ell = 0$) and the first and second *modified* KP hierarchies ($\ell = 1, 2$):

$$\frac{\partial L}{\partial t_m} + \Big[(L^m)_{\geq \ell} \, , \, L \Big] = 0 \tag{49}$$

The three KP hierarchies are related through symplectic gauge transformations [35]. Here we shall concentrate on various *Poisson reductions* [2] of the standard KP hierarchy which, in particular, appear in the context of matrix models of strings.

The phase space of the standard KP integrable system is:

$$\mathcal{M}_{KP} = \Big\{ L = D + \sum_{k=1}^{\infty} u_k(x) D^{-k} \Big\} \tag{50}$$

It is a trivial Poisson reduction from $\mathcal{M} = \Psi\mathcal{DO}^* \simeq \Psi\mathcal{DO}$). The KK Poisson brackets (first eq.(41)) read on (50):

$$\{ u_n(x) \, , \, u_m(y) \} = \Omega_{n-1,m-1}(u(x)) \, \delta(x-y) \; , \tag{51}$$

$$\Omega_{nm}(u(x)) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} u_{n+m-k+1}(x) D_x^k - \sum_{k=0}^{m} \binom{m}{k} D_x^k u_{n+m-k+1}(x)$$

and are recognized as the centerless $W_{1+\infty}$ algebra [9], which is isomorphic to the algebra of differential operators on the circle $\mathcal{DOP}(S^1) \subset \Psi\mathcal{DO}$ [37].

Let us go back to the flow eqs. in (multi)matrix models (24) with Lax operator (27). One can show [38] that the space of $2q$-boson Lax operators:

$$\mathcal{M}_q = \Big\{ L_q = D + \sum_{l=1}^{q} A_l \left(D - B_l\right)^{-1} \left(D - B_{l+1}\right)^{-1} \ldots \left(D - B_q\right)^{-1} \Big\} \tag{52}$$

is a *Poisson reduction* of the full KP hierarchy given by (50). Therefore, the flow eqs. (24) are Hamiltonian and completely integrable. In [38] the Darboux canonical pairs of coordinates $\big(a_r(x), b_r(x)\big), r = 1, .., q$ were found:

$$\{ a_r(x), \, b_s(y) \}_{P'} = -\delta_{rs} \partial_x \delta(x-y) \tag{53}$$

$$B_l = b_l + b_{l+1} + \cdots + b_q \quad , \quad A_q = a_q \tag{54}$$

$$A_{q-r}(a,b) = \sum_{n_r=r}^{q-1} \cdots \sum_{n_2=2}^{n_3-1} \sum_{n_1=1}^{n_2-1} \left(\partial + b_{n_r} + \cdots + b_{n_r-r+1}\right) \cdots \left(\partial + b_{n_1}\right) a_{n_1}$$

By expressing the $2q$-boson Lax operator (52) as a power series in D^{-1} it could be rewritten in the form (50):

[2] The general notions could be found in [36]. Let $(\mathcal{M}, P)$ be a smooth Poisson manifold with Poisson structure $P \; : \; T^*(\mathcal{M}) \longrightarrow T(\mathcal{M})$ and let S be a smooth submanifold of $\mathcal{M}$ with an embedding $\mu \; : \; S \longrightarrow \mathcal{M}$. A Poisson structure $P' \; : \; T^*(S) \longrightarrow T(S)$ on $S \subset \mathcal{M}$ is called *Poisson reduction* of P if for an arbitrary pair of functions on $\mathcal{M}$ the following property is satisfied : $\mu^* \left(\{f_1, \, f_2\}_P \right) = \{\mu^* f_1, \, \mu^* f_2\}_{P'}$.

$$L_q = D + \sum_{k=1}^{\infty} U_k[(a,b)](x) D^{-k} \tag{55}$$

$$U_k[(a,b)](x) = a_q P^{(1)}_{k-1}(b_q) \tag{56}$$
$$+ \sum_{r=1}^{\min(q-1,k-1)} A_{q-r}(a,b) P^{(r+1)}_{k-1-r}(b_q, b_q + b_{q-1}, \ldots, \sum_{l=q-r}^{q} b_l)$$

where $A_{q-r}(a,b)$ are the same as in (54), and $P_n^{(N)}$ denote the multiple Faá di Bruno polynomials:

$$P_n^{(N)}(B_N, \ldots, B_1) = \sum_{m_1+\cdots+m_N=n} (-\partial + B_1)^{m_1} \cdots (-\partial + B_N)^{m_N} \cdot 1 \tag{57}$$

Thus, the Poisson reduction described above leads to the construction of a series of $2q$-boson representations, $q = 1, 2, ..,$ of $W_{1+\infty}$ algebra in terms of Poisson brackets for the functions (56), as follows from (51):

$$\Big\{ U_k[(a,b)](x)\, ,\, U_l[(a,b)](y) \Big\} = \Omega_{k-1,l-1}(U[(a,b)])\delta(x-y) \tag{58}$$

Until now only the first Hamiltonian structure for integrable systems in the AKS-RS scheme has been discussed. It is given by the KK bracket (35) on $\mathcal{G}_R^*$.

Completely integrable systems possess another remarkable feature – that of *multi*-Hamiltonian structure, *i.e.*, they always possess at least two independent *compatible* Poisson structures [3].

The second Hamiltonian structure for general Lax operators (50) has the form[4]:

$$\{ \langle L|\, X \rangle \, , \langle L|\, Y \rangle \} = Tr_A \Big((LX)_+ \, LY - (XL)_+ \, YL \Big)$$
$$+ \int dx\, Res\Big([\, L\, ,\, X\,] \Big) \partial^{-1} Res\Big([\, L\, ,\, Y\,] \Big) \tag{59}$$

Here Tr_A denotes the Adler trace (44) and the subscript $+$ indicates taking the purely differential part. For the coefficient fields $u_k(x)$ of L (50) the second KP Poisson algebra (59) yields the *nonlinear* (*i.e.*, non-Lie) $\widehat{W}_\infty$ algebra [42]. The latter appears as a unique (modulo certain homogeneity assumptions) nonlinear deformation of $W_{1+\infty}$ algebra. The Darboux variables for (59) for the operators L_q (52) can also be constructed (see next subsection). The second KP Hamiltonian structure can be understood as a *Lie-Poisson* structure on the Lie-Poisson group of purely integral operators – the Volterra group.

[3] Two Poisson structures $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ are called *compatible* if their linear combination $\{\cdot,\cdot\}_\lambda = \{\cdot,\cdot\}_1 + \lambda\{\cdot,\cdot\}_2$ is also a Poisson structure.

[4] The second term on the r.h.s. of (59) is a Dirac bracket term due to the second class constraint $u_0 = 0$ in L (50).

3.3 Lie-Poisson Groups and Lie Bi-Algebras: Hamiltonian approach

The notion of a Lie-Poisson group was introduced by Drinfeld [16]. A Lie group G is called *Lie-Poisson* if there exists a Poisson structure on the algebra of smooth functions on it $Fun(G)$ which is compatible with the group multiplication, *i.e.*, $\Delta\left(\left\{ F_1 \, , \, F_2 \right\}\right) = \left\{ \Delta\left(F_1\right) , \Delta\left(F_2\right) \right\}$, where Δ denotes the coproduct in $Fun(G)$ $\left(\Delta F(g,h) = F(gh) \right)$. Poisson structure with this property is called Lie-Poisson and is given by:

$$\left\{ F_1(g) \, , \, F_2(g) \right\} = \left\langle \nabla_L F_1(g) \otimes \nabla_L F_2(g) \middle| r(g) \right\rangle \\ = - \left\langle \nabla_R F_1(g) \otimes \nabla_R F_2(g) \middle| r(g^{-1}) \right\rangle \tag{60}$$

$$\left\langle \nabla_L F(g) \middle| X \right\rangle = \frac{d}{dt}\Big|_{t=0} F\left(e^{tX} g\right) \; , \; \left\langle \nabla_R F(g) \middle| X \right\rangle = \frac{d}{dt}\Big|_{t=0} F\left(g e^{tX}\right) \tag{61}$$

where $\nabla_{L,R}$ denote left, right Lie-derivatives, and $r(g)$ is a *cocycle* on G with values in $\mathcal{G} \otimes \mathcal{G}$:

$$r(gh) = r(g) + Ad(g) \otimes Ad(g)\, r(h) \tag{62}$$

In the case when $r(g)$ is a *coboundary*:

$$r(g) = Ad(g) \otimes Ad(g)\, r_0 - r_0 \tag{63}$$

with $r_0 \in \mathcal{G} \otimes \mathcal{G}$ being a constant element, one finds that the Jacobi identity for (60) reduces precisely to the YBE (34) for r_0 as a classical r-matrix.

For matrix groups eq.(60) can be written in a simpler form:

$$\{g \overset{\otimes}{,} g\} = r(g)\, g \otimes g \quad \Big(\; \{g \overset{\otimes}{,} g\} = - \left[r_0 \, , \, g \otimes g \right] \;\; in\ the\ case\ of\ (63) \Big) \tag{64}$$

The cocycle condition (62) implies the following exterior derivative equation:

$$d\, r(g) = \left[Y(g) \otimes 1 + 1 \otimes Y(g) \, , \, r(g) \right] - \phi\left(Y(g)\right) , \tag{65}$$

$$\phi(X) = - \frac{d}{dt}\Big|_{t=0} r\left(e^{Xt}\right) \quad for \quad \forall X \in \mathcal{G} \tag{66}$$

where $Y(g) = dg\, g^{-1}$ denotes the Maurer-Cartan one-form on $\mathcal{G}$, and ϕ defined in (66) is a $\mathcal{G} \otimes \mathcal{G}$-valued cocycle on $\mathcal{G}$:

$$\phi\left([X \, , \, Y]\right) = \left[X \otimes 1 + 1 \otimes X \, , \, \phi(Y) \right] + \left[\phi(X) \, , \, Y \otimes 1 + 1 \otimes Y \right] \tag{67}$$

The solution of (65) is a coboundary $r(g)$ (63) if and only if ϕ is a coboundary:

(68) $$\phi(X) = [r_0\,,\, X \otimes 1 + 1 \otimes X]$$

The cocycle ϕ allows to introduce a Lie-commutator $[\cdot,\cdot]_*$ on the dual space $\mathcal{G}^*$ and, correspondingly, coadjoint action $ad_*^*(\cdot)$ of $\mathcal{G}^*$ on $\mathcal{G}$ as follows:

$$\Big\langle \big[\,U\,,\,V\,\big]_* \Big| X \Big\rangle \equiv -\Big\langle V \Big| ad_*^*(U)X \Big\rangle = \Big\langle U \otimes V \Big| \phi(X) \Big\rangle\,, \quad U,V \in \mathcal{G}^*, \quad X \in \mathcal{G}$$
(69)

whereas the mixed commutator between elements of $\mathcal{G}$ and $\mathcal{G}^*$ is defined as:

(70) $$[\,X\,,\,U\,] = ad^*(X)U - ad_*^*(U)X$$

An important theorem by Drinfeld [16] states that the Lie algebra $\mathcal{G}$ of each Lie-Poisson group G is a Lie-*bialgebra* and *vice versa*. Thus, by means of (69) and (70) the direct sum (as vector space) $\mathcal{D} = \mathcal{G} \oplus \mathcal{G}^*$ becomes itself a Lie algebra called the *double*, such that $\mathcal{G}$ and $\mathcal{G}^*$ are isotropic subalgebras of $\mathcal{D}$ w.r.t. the Killing form on $\mathcal{D}$:

(71) $$\Big((X_1,U_1)\,,\,(X_2,U_2)\Big) = \langle U_1|\, X_2\rangle + \langle U_2|\, X_1\rangle \quad \forall\,(X_{1,2},U_{1,2}) \in \mathcal{D}$$

The triple $\left(\mathcal{D},\mathcal{G},\mathcal{G}^*\right)$ is also called *Manin triple*. The *double group* $\widetilde{D} \simeq G \times G^*$, corresponding to $\mathcal{D}$, is a direct product (as a manifold) of its subgroups G and G^* corresponding to $\mathcal{G}$ and $\mathcal{G}^*$, respectively. The above construction is symmetric under the replacements $\mathcal{G} \longleftrightarrow \mathcal{G}^*\,,\; G \longleftrightarrow G^*$.

The explicit solution for the Lie-Poisson group cocycle $r(g)$ (62) reads:

(72) $$\langle U \otimes V|\, r(g)\rangle = \Big\langle \left(g^{-1}Vg\right)_- \Big| \left(g^{-1}Ug\right)_+ \Big\rangle \quad \forall U,V \in \mathcal{G}^*$$

where the subscripts $(\pm)$ indicate projections in the double algebra $\mathcal{D}$ along $\mathcal{G}\,,\,\mathcal{G}^*$, respectively.

As in the AKS-RS Lie-algebraic scheme, one can construct integrable Hamiltonian systems on Lie-Poisson groups. Indeed, from (62) or (63) one can easily verify that all $Ad(\cdot)$-invariant functions on G, $H[hgh^{-1}] = H[g]$, are in involution w.r.t. the Lie-Poisson structure (60): $\{\,H_k[g]\,,\,H_l[g]\,\} = 0$. Similarly, the analogues of the Hamiltonian eqs. of motion (36) and the associated geometric action (37) take the form:

(73) $$\frac{\partial g}{\partial t_k} g^{-1} = -\hat{r}_g\left(\nabla_L H_k[g]\right)$$

(74) $$\mathcal{W}[g] = -\frac{1}{2}\int d^{-1}\left(\Big\langle \hat{s}_g(Y(g)) \Big| Y(g) \Big\rangle\right) - \int dt\, H_k[g]$$

where the action of the operator $\hat{r}_g : \mathcal{G}^* \longrightarrow \mathcal{G}$ is defined by $\langle U | \, \hat{r}_g (V) \rangle \equiv \langle U \otimes V | \, r(g) \rangle$, cf. (72), $\hat{s}_g = \hat{r}_g^{-1}$, and d^{-1} denotes inverse operator of the exterior derivative defined on the closed forms [5].

An example of this construction, worked out in [39], is provided by the extended Volterra algebra of purely pseudo-differential operators (c is an arbitrary constant):

$$\mathcal{G} \equiv \left(\Psi\widetilde{\mathcal{DO}}\right)_- = \left\{ \sum_{k \geq 1} u_k(x) D^{-k} + c \ln D \right\} \tag{75}$$

Its dual is the extended algebra of differential operators (α an arbitrary constant):

$$\mathcal{G}^* \equiv \widetilde{\mathcal{DOP}} \simeq W_{1+\infty} = \left\{ \sum_{l \geq 0} D^l v_l(x) + \alpha \hat{E} \right\} , \tag{76}$$

and the Lie-double is the extended algebra of all pseudo-differential operators:

$$\mathcal{D} \equiv \Psi\widetilde{\mathcal{DO}} = \left(\Psi\widetilde{\mathcal{DO}}\right)_- \oplus \widetilde{\mathcal{DOP}} \tag{77}$$

Here $\hat{E}$ indicates the central element of $W_{1+\infty}$, as well as of the whole $\Psi\widetilde{\mathcal{DO}}$, which is dual to $\ln D$, cf. [37, 39]. The corresponding extended Volterra group (exponentiation of (75)):

$$G \equiv \left(\Psi\widetilde{DO}\right)_- = \left\{ g \equiv L = \left(1 + \sum_{k \geq 1} \tilde{u}_k(x) D^{-k}\right) \circ D^c \right\} \tag{78}$$

can be viewed as a set of spaces (for each fixed c) of Lax operators of generalized KP hierarchies. (The KP hierarchy (49) is recovered for $c = 1$). In [39] it was shown that the second Hamiltonian structure (59) coincides with the Lie-Poisson structure (64) with the cocycle $r(g)$ (72) for the group $G = \left(\Psi\widetilde{DO}\right)_-$ (78).

Let us go back to the example of $2q$-boson KP Lax operators appearing in the multi-matrix string models (52). In analogy with eqs.(54)–(53) one can express [41] the coefficient fields $(A_l, B_l)_{l=1}^q$ of L_q (52) in terms of Darboux canonical pairs of fields $(c_r, e_r)_{r=1}^q$ w.r.t. the second KP Hamiltonian structure (59):

$$\{c_k(x) \, , \, e_l(y)\} = -\delta_{kl} \partial_x \delta(x-y) \; , \quad k,l = 1,2,\ldots,q \tag{79}$$

$$B_k = e_k + \sum_{l=k}^{q} c_l \; , \qquad 1 \leq k \leq q \; ; \qquad A_q = \sum_{r=1}^{q} (\partial + c_r) \, e_r \tag{80}$$

[5] Closeness follows from the exterior derivative equation: $d\hat{s}_g = ad^*\big(Y(g)\big)\hat{s}_g - \hat{s}_g ad\big(Y(g)\big) - \hat{s}_g \phi\big(Y(g)\big)\hat{s}_g$, satisfied by the inverse cocycle operator $\hat{s}_g$ as a consequence of eq.(65).

$$
\begin{aligned}
(81) \qquad A_k = \sum_{n_k=1}^{k} & \left(\partial + e_{n_k} - e_{n_k+q-k} + \sum_{l_k=n_k}^{n_k+q-k} c_{l_k}\right) \\
\times \sum_{n_{k-1}=1}^{n_k} & \left(\partial + e_{n_{k-1}} - e_{n_{k-1}+q-1-k} + \sum_{l_{k-1}=n_{k-1}}^{n_{k-1}+q-1-k} c_{l_{k-1}}\right) \times \cdots \\
\times \sum_{n_2=1}^{n_3} \left(\partial + e_{n_2} - e_{n_2+1} + c_{n_2} + c_{n_2+1}\right) & \sum_{n_1=1}^{n_2} \left(\partial + c_{n_1}\right) e_{n_1}, \quad 1 \leq k \leq q-1
\end{aligned}
$$

These equations are equivalent to the following "dressing" form for the $2q$-boson KP Lax operator (52):

$$
L_q = \mathcal{U}_q \ldots \mathcal{U}_1 \, D \, \mathcal{V}_1^{-1} \ldots \mathcal{V}_q^{-1} \tag{82}
$$

$$
\mathcal{U}_k \equiv (D - e_k) \, e^{\int c_k} \quad , \quad \mathcal{V}_k \equiv e^{\int c_k} (D - e_k) \quad , \quad k = 1, \ldots, q \tag{83}
$$

Eqs.(79)–(81) or, equivalently, eqs.(82)–(83) can be viewed as *generalized Miura transformation* for the $2q$-boson KP hierarchy [6]. The Miura form of L_q (82) reads explicitly:

$$
L_q = D + \sum_{k=1}^{\infty} U_k[(c,e)](x) D^{-k} \tag{84}
$$

$$
\begin{aligned}
(85) \qquad U_k[(c,e)](x) = P_{k-1}^{(1)} \left(e_q + c_q\right) \sum_{l=1}^{q} \left(\partial + c_l\right) e_l + \\
\sum_{r=1}^{\min(q-1,k-1)} A_{q-r}(c,e) P_{k-1-r}^{(r+1)} (e_q + c_q, e_{q-1} + c_{q-1} + c_q, \ldots, e_{q-r} + \sum_{l=q-r}^{q} c_l)
\end{aligned}
$$

where $A_{q-r}(c,e)$ are the same as in (81), and $P_n^{(N)}$ denote the (multiple) Faá di Bruno polynomials (57).

Now, in complete analogy with eqs.(55)–(58), which yield a series of realizations of the linear $W_{1+\infty}$ algebra in terms of $2q$ bosons, one obtains, after substitution of (84)–(85) into (59), a series of explicit Poisson bracket realizations of the nonlinear $\widehat{W}_\infty$ algebra in terms of $2q$ bosonic fields for any $q = 1, 2, \ldots$. This algebra plays an important rôle as a "hidden" symmetry algebra in string-theory-inspired models with black hole solutions [42].

Concluding this section, let us note that in the general case Lie-Poisson groups provide a natural geometric description of the *dressing symmetries* in completely integrable models [43, 44]. The Lie-Poisson structures (64) also appear in the context of the classical inverse scattering method [5] as fundamental *Sklyanin brackets* for the monodromy matrix $g \simeq T(\lambda)$ of the auxiliary linear spectral problem.

[6]For discussion of the generalized Miura transformation and the associated Kuperschmidt-Wilson theorem, we refer to [40].

4 Quantum Integrable Models

4.1 Quantization of Lie-Poisson Groups: Quantum Groups

The quantization of completely integrable models with a Hamiltonian structure which is Lie-Poisson, lead historically to the first explicit construction by the quantum scattering method [11], of *quantum groups*. These were subsequently identified with *quasi-triangular Hopf algebras* [13].

Among the various ways to introduce quantum groups there exists an approach [12], whose conceptual point of view underscores both the quantum mechanical as well as the Hopf algebraic aspects in the quantization of Lie-Poisson groups. On the one hand, $Fun(G)$ can be viewed as an Abelian associative algebra of "observables" of a classical Hamiltonian system $(\mathcal{M}, \mathcal{P})$ with a phase space $\mathcal{M} = G$ and Poisson structure which is Lie-Poisson, *i.e.*, $\mathcal{P} = \mathcal{P}_{LP}$ given by (60):

$$\mathcal{P}_{LP}\,(F_1, F_2) \equiv \{\, F_1\,,\, F_2\,\} = \langle \nabla_L F_1 \otimes \nabla_L F_2 - \nabla_R F_1 \otimes \nabla_R F_2 |\, r_0 \rangle\,, \tag{86}$$

where the classical r-matrix r_0 satisfies the classical YBE (34). From now on we shall consider only coboundary Lie-Poisson groups (63). On the other hand, one can check that $Fun(G)$ is endowed with a structure of a commutative and non-cocommutative Hopf algebra $\mathcal{A}_0(m, \Delta, S, \varepsilon) \equiv Fun(G)$ with a product $m\,(F_1, F_2)\,(g) = F_1(g)F_2(g)$, coproduct $\Delta(F)(g_1, g_2) = F(g_1 g_2)$, antipode $SF(g) = F(g^{-1})$ and counit $\varepsilon(F) = F(e)$, and this Hopf structure is compatible with the Poisson structure (86), *i.e.*, $\Delta \circ \mathcal{P} = \mathcal{P} \circ \Delta$.

Thus, the quantization of a Lie-Poisson group G may be viewed as a generalization of Weyl quantization: $Fun(G) \longrightarrow Fun_h(G)$ of a classical Hamiltonian system defined by $(\mathcal{M} \equiv G, \mathcal{P}_{LP})$, *i.e.*, it is a *non-commutative* deformation of the product $m \longrightarrow m_h$ with a deformation parameter h, which satisfies the additional condition that the deformed algebra $Fun_h(G) \equiv \mathcal{A}_h(m_h, \Delta, S, \varepsilon)$ is a non-commutative and non-cocommutative Hopf algebra.

Let us recall [17], that Weyl quantization for a Hamiltonian system defined on a Poisson manifold $(\mathcal{M}, \mathcal{P})$ with local coordinates (x^i) and constant Poisson tensor $\{\, x^i\,,\, x^j\,\} = P^{ij}$ is given by the associative and non-commutative *Moyal* product:

$$\begin{aligned} m_h\,(F_1, F_2) &\equiv F_1 \star_h F_2 = m \circ e^{\frac{h}{2}\mathcal{P}}\,(F_1, F_2) \\ &= F_1 \cdot F_2 + \frac{h}{2}\{\, F_1\,,\, F_2\,\} + O(h^2) \end{aligned} \tag{87}$$

$$\mathcal{P} = P^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}\ :\ Fun(\mathcal{M}) \otimes Fun(\mathcal{M}) \longrightarrow Fun(\mathcal{M}) \otimes Fun(\mathcal{M})$$

Here $\mathcal{P}$ is the Poisson bi-vector field. Note, that the form of the first order term in the h-expansion (last eq.(87)) is dictated by the *semiclassical correspondence principle*.

For Lie-Poisson groups $(G, \mathcal{P}_{LP})$, the deformed product m_h which defines the deformed Hopf algebra structure and satisfies the semiclassical condition, can be constructed as follows [12]. Let us choose a basis $\{X^i\}$ in $\mathcal{U}(\mathcal{G})$ – the universal enveloping algebra of the Lie algebra $\mathcal{G}$ of G, and let $\pi_{L,R}$ denote the representations of $\mathcal{U}(\mathcal{G})$ in terms of left, right Lie derivatives: $\pi_{L,R}(X^i) = \nabla^i_{L,R}$ (see eq.(61)). Then:

$$
\begin{aligned}
&(88) \qquad m_h = m \circ \widetilde{\Lambda} \quad , \quad \widetilde{\Lambda} = (\pi_L \otimes \pi_L)(\Lambda) \circ (\pi_R \otimes \pi_R)(\Lambda^{-1}) \\
&(89) \qquad \Lambda(X,Y) = \sum_{\{\alpha\},\{\beta\}} c_{\{\alpha\},\{\beta\}}(h) \prod_{i=1}^{dim\,\mathcal{G}} \left(X^i\right)^{\alpha_i} \prod_{j=1}^{dim\,\mathcal{G}} \left(Y^j\right)^{\beta_j} \\
&\qquad\qquad = 1 + \frac{h}{2} r_{ij} X^i Y^j + O(h^2)
\end{aligned}
$$

with the following notations. The coefficients in $\Lambda \; : \; \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G}) \longrightarrow \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G})[[h]]$ are power series in h ; $\{X^i\}$ and $\{Y^j\}$ are generator basises in the first and second copy of $\mathcal{U}(\mathcal{G})$, respectively; $\|r_{ij}\| = r_0$ is just the classical r-matrix satisfying (34). The associativity condition for m_h (88) implies the following quadratic equation for Λ on $\mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G})[[h]]$ (X, Y, Z below correspond to the first, second and third factor $\mathcal{U}(\mathcal{G})$ in the tensor product):

$$
\begin{aligned}
&(90) \qquad \Lambda(X+Y,Z)\Lambda(X,Y) = \Lambda(X,Y+Z)\Lambda(Y,Z) \\
&\qquad\qquad \Lambda(X,0) = \Lambda(0,Y) = 1
\end{aligned}
$$

Defining $\bar{R}(X,Y) = \Lambda^{-1}(Y,X)\Lambda(X,Y) \in \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G})[[h]]$, one obtains from (90):

$$
\begin{aligned}
&(91) \qquad \bar{R}(X,Y)\bar{R}(X,Z)\bar{R}(Y,Z) = \bar{R}(Y,Z)\bar{R}(X,Z)\bar{R}(X,Y) \\
&\qquad \bar{R}(X,Y)\bar{R}(Y,X) = 1 \quad ; \quad \bar{R}(X,Y) = 1 + h\, r_{ij} X^i Y^j + O(h^2)
\end{aligned}
$$

$\bar{R}$ is called *universal quantum R-matrix* associated with the classical r-matrix r_0. If $\rho \; : \; \mathcal{G} \longrightarrow End(\mathcal{V})$ is some representation of $\mathcal{G}$ in a finite-dimensional vector space $\mathcal{V}$, then the matrix $R = (\rho \otimes \rho)\left(\bar{R}\right) \in End(\mathcal{V} \otimes \mathcal{V})$ satisfies the *quantum Yang-Baxter* equation (QYBE) and the "unitarity" condition, ($P \in End(\mathcal{V} \otimes \mathcal{V})$ being the permutation operator $PX \otimes Y = Y \otimes X$):

$$
(92) \qquad \overset{12}{R}\,\overset{13}{R}\,\overset{23}{R} = \overset{23}{R}\,\overset{13}{R}\,\overset{12}{R} \quad ; \quad RPRP = 1 \quad ; \quad R = 1 + h\, r_0 + O(h^2)
$$

The indices $12, 13, 23$ indicate the various embeddings of $R \in End(\mathcal{V} \otimes \mathcal{V})$ in $End(\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V})$.

In particular, when G is a matrix group and the functions F_i are chosen to be the matrix elements of $g \in G$, $F_1(g) = g_{ab}$, $F_2(g) = g_{cd}$, one obtains from the deformed product (88) and using matrix tensor notations:

$$R\overset{1}{g}\overset{2}{g}=\overset{2}{g}\overset{1}{g}R \quad , \qquad \overset{1}{g}=(g\otimes 1) \quad , \quad \overset{2}{g}=(1\otimes g) \tag{93}$$

The semiclassical limit of (93) is precisely given by the Lie-Poisson bracket (64). Eq.(93) is nothing but the fundamental commutation relations for the matrix elements of the quantum monodromy matrix (with suppressed dependence on the spectral parameter) in the quantum inverse scattering method [11].

Various treatments and numerous applications of QYBE could be found in ref.[45]. For an incomplete list of parallel developments in abstract Hopf algebra approach, the reader is referred to [46].

4.2 Soliton Scattering in Completely Integrable Models

Let us consider a completely integrable $D=2$ relativistic field theory defined by the action $S[\phi]=\int d^2x\,\mathcal{L}(\phi,\partial\phi)$ which is assumed to be a local functional of the fundamental fields (collectively denoted by ϕ) and their derivatives. It is useful to introduce the light-cone coordinates: $x^{\pm}=\frac{1}{2}(x^1\pm x^0)$. Complete integrability implies the existence of an infinite number of independent integrals of motion in involution $Q^{(s)}$, whose densities are local (as functionals of ϕ and its derivatives) conserved currents:

$$\begin{aligned} Q^{(s)} &= \oint\left(T^{(s+1)}dx^{-}+\Theta^{(s-1)}dx^{+}\right), \quad s=1,2,\ldots \quad ; \\ \partial_{+}T^{(s+1)} &= \partial_{-}\Theta^{(s-1)} \end{aligned} \tag{94}$$

here s indicates the $D=2$ Lorentz weight. Thus, if the property of complete integrability survives after quantization, it should imply, in particular, that the quantum *renormalized* Ward identities for the *renormalized* quantum conserved currents $\left(\widetilde{T}^{(s+1)},\widetilde{\Theta}^{(s-1)}\right)$ should be satisfied:

$$\begin{aligned} \partial_{+}\left\langle\widetilde{T}^{(s+1)}(x)\phi(x_1)\ldots\phi(x_n)\right\rangle - \partial_{-}\left\langle\widetilde{\Theta}^{(s-1)}(x)\phi(x_1)\ldots\phi(x_n)\right\rangle \\ = \delta - function\ terms \end{aligned} \tag{95}$$

Here $\langle\ldots\rangle$ denote the time-ordered correlation functions. The infinite set of Ward identities leads to severe restrictions on the particle ("soliton") scattering processes – conservation of all (odd) powers of momenta of incoming and outgoing particles:

$$\sum_{l=1}^{N_{in}} p_{l(in)}^{2n+1} = \sum_{l=1}^{N_{out}} p_{l(out)}^{2n+1} \tag{96}$$

with N_{in}, N_{out} denoting the number of incoming, outgoing particles which, in turn, implies [47]:

1. no multi-particle production, *i.e.*, $N_{in} = N_{out}$;
2. factorization of multi-particle scattering amplitudes.

The last property leads to the remarkable Zamolodchikov's factorization eqs. for the 3-particle amplitudes [6], meaning that any 3-particle scattering process is accomplished as a sequence of 2-particle scatterings only and, moreover, the amplitude does not depend on the order in which these sequential 2-particle scatterings occur:

$$(97)\quad S^{k_1k_2}_{i_1i_2}(\theta_{12})\, S^{j_1k_3}_{k_1i_3}(\theta_{13})\, S^{j_2j_3}_{k_2k_3}(\theta_{23}) = S^{k_2k_3}_{i_2i_3}(\theta_{23})\, S^{k_1j_3}_{i_1k_3}(\theta_{13})\, S^{j_1j_2}_{k_1k_2}(\theta_{12})$$

with $\theta_{ab} \equiv \theta_a - \theta_b$, $a, b = 1, 2, 3$. In (97) the following notations are used: $S^{kl}_{ij}(\theta_{12})$ denotes 2-particle scattering amplitude of incoming particles of "type" labelled by i and j into outgoing particles of type k and l and (on-mass-shell) momenta $p_a = m_a(\cosh\theta_a, \sinh\theta_a)$, $a = 1, 2$ (θ_a is the relativistic rapidity).

Now, denoting by $\mathcal{V}$ the vector space of internal particle symmetry (particle "types"), one can regard the matrix of the 2-particle amplitude as:

$$(98)\quad S(\theta_{12}) = \| S^{kl}_{ij}(\theta_{12}) \| \in Mat(\mathcal{V}) \otimes Mat(\mathcal{V}) \quad (for\,fixed\,\theta_{12})$$
$$\overset{12}{S}(\theta_{12}) \equiv S(\theta_{12}) \otimes 1 \in Mat(\mathcal{V}) \otimes Mat(\mathcal{V}) \otimes Mat(\mathcal{V})$$

and, accordingly, for $\overset{13}{S}(\theta_{13})$ and $\overset{23}{S}(\theta_{23})$. Then it is straightforward to identify (97) with the QYBE (92) in the quantum group framework.

A realization of the matrix quantum group relation (93) is provided by the symmetries of the soliton scattering states [49]. Let $T_{ij}(\theta)$ is an internal symmetry operator acting on the 1-particle asymptotic space: $|(\theta, i)\rangle \longrightarrow T_{ij}(\theta)|(\theta, j)\rangle$; $|(\theta, i)\rangle \in \mathcal{V}$ and $\|T_{ij}(\theta)\| \in Mat(\mathcal{V})$.

As a result of the integrability the 2-particle S-matrix:

$$(99)\qquad |\,(\theta_1, i_1)\,,\,(\theta_2, i_2)\rangle^{in} = S^{j_1j_2}_{i_1i_2}(\theta_{12})\,|\,(\theta_2, j_2)\,,\,(\theta_1, j_1)\rangle^{out}$$

can be viewed (for fixed "rapidities") as a mapping $\overset{12}{S} : \overset{1}{\mathcal{V}} \otimes \overset{2}{\mathcal{V}} \longrightarrow \overset{2}{\mathcal{V}} \otimes \overset{1}{\mathcal{V}}$. Its invariance under the asymptotic states's symmetry on the subspace of 2-particle asymptotic states gives (in the notation of eqn. (93)):

$$(100)\qquad \overset{12}{S}(\theta_{12})\,\overset{1}{T}(\theta_1)\,\overset{2}{T}(\theta_2) = \overset{2}{T}(\theta_2)\,\overset{1}{T}(\theta_1)\,\overset{12}{S}(\theta_{12})$$

This is straightforwardly identified with the structural relations (93) for quantum groups.

As pointed out in [50] quantum group relations of the form (97) and (100) appear in exactly solvable lattice models of planar statistical mechanics, but in this case – with purely imaginary "rapidities" ($\theta = i\alpha$, α being angle characterizing the rectangular lattice), $S^{kl}_{ij}(\alpha)$ being the matrix of Boltzmann weights at each lattice vertex, and $T_{ij}(\alpha)$ denoting the row transfer matrix.

4.3 Quantum Field Theory Approach to Integrable Models with Dynamically Broken Conformal Invariance

Finally, let us briefly discuss the construction of higher local quantum conserved currents satisfying the Ward identities (95), which is the heart of the quantum field theory approach to quantization of completely integrable models. Recently, Zamolodchikov [14] proposed a beautiful general formalism based on treating integrable models as *mass perturbations* of conformal field theories: $S[\phi] = S_{conf}[\phi] + \sum_i m_i \int d^2x\, B_i(\phi, \partial\phi)$, where the coupling constants m_i have positive mass dimensions and $B_i(\phi, \partial\phi)$ are composite fields with conformal dimensions less than 2. Zamolodchikov's approach is purely algebraic since in general it is not possible to explicitly find expressions for $S_{conf}[\phi]$ and $B_i(\phi, \partial\phi)$ as local functionals of the local fundamental fields $\{\phi\}$. It uses exact results from the representation theory of the Virasoro algebra, in particular, information about the spectrum of conformal field dimensions.

There exist, however, interesting classes of $D = 2$ integrable field theories, which are conformally invariant on the classical level but, upon quantization, undergo *dimensional transmutation*, manifested through dynamical *mass generation*. This leads to anomalous conformal symmetry breaking. An example is the $O(N)$ nonlinear sigma-model and its supersymmetric generalization defined by:

$$\text{(101)} \qquad \mathcal{L}_{NL\sigma} = \frac{1}{2}\partial_+ n^a \partial_- n_a, \qquad \vec{n}^2 = N/g, \qquad \vec{n} = \left(n^1, \ldots n^N\right)$$

$$\text{(102)} \; \mathcal{L}^{sysy}_{NL\sigma} = \frac{1}{2}\partial_+ n^a \partial_- n_a + i\bar{\psi}^a \gamma^\mu \partial_\mu \psi_a - \frac{g}{N}\left(\bar{\psi}^a \psi_a\right)^2, \quad n^a \psi_a = 0$$

The dimensional transmutation phenomenon clearly precludes the use of conformal perturbation approach to (101), (102). However, there is an alternative nonperturbative [7] treatment of quantum field theory models with $O(N)$ or $SU(N)$ internal symmetry – the $1/N$ expansion [51].

Let us briefly illustrate the construction [52] of higher local quantum conserved currents for (101) [53] within the $1/N$ expansion framework (the same techniques applies to other $1/N$-expandable integrable models as well). The $1/N$ expansion is obtained from the generating functional of the time-ordered correlation functions:

$$\text{(103)} Z[J] = \int \mathcal{D}\vec{n} \prod_x \delta\left(\vec{n}^2 - N/g\right) \exp\left\{ i \int d^2x \left[\frac{1}{2}(\partial\vec{n})^2 + \left(\vec{J}, \vec{n}\right)\right]\right\}$$

[7] The term "nonperturbative" refers to expansions different from (or, *e.g.*, partial resummations of) the ordinary perturbation theory w.r.t. the coupling constant g in (101) and (102) which is plagued by infrared divergences in $D = 2$.

$$= \int \mathcal{D}\sigma \, \exp\left\{ -\frac{N}{2} S_1[\sigma] + \frac{i}{2} \int d^2x \, d^2y \, \left(\vec{J}(x), (-\partial^2 + \sigma)^{-1} \vec{J}(y) \right) \right\}$$

$$S_1[\sigma] \equiv Tr \ln(-\partial^2 + \sigma) + \frac{i}{g} \int d^2x \, \sigma \tag{104}$$

by expanding the effective σ-field action (104) around its stationary point $\sigma_c \equiv m^2 = \mu^2 e^{-4\pi/g}$ (dynamically generated mass of the "Goldstone" field $\vec{n}$, μ being the renormalization scale), *i.e.*, $\sigma(x) = m^2 + \frac{1}{\sqrt{N}} \tilde{\sigma}(x)$. As a result, one arrives at the $1/N$ diagram technique with (free) propagators in momentum space:

$$\left\langle n^a \, n^b \right\rangle_{(0)} = -i \left(m^2 + p^2 \right)^{-1} \delta^{ab}, \qquad \langle \tilde{\sigma} \, \tilde{\sigma} \rangle_{(0)} = \left(\Sigma \left(p^2 \right) \right)^{-1} \tag{105}$$

$$\Sigma \left(p^2 \right) = \int \frac{d^2k}{(2\pi)^2} \frac{1}{(m^2 + k^2)\,(m^2 + (p-k)^2)}$$

and tri-linear $\tilde{\sigma}nn$-vertices.

The $1/N$ expansion can be renormalized [54] by adapting the *BPHZ* [8] renormalization technique. A remarkable property of the $1/N$ expansion is that the nonlinearity of the "Goldstone" field $\vec{n}(x)$ is preserved on the quantum level as an identity on the correlation functions, in spite of the manifest linear $O(N)$ symmetry of (105):

$$\left\langle \mathcal{N}\left[\vec{n}^2 P(\vec{n}, \partial\vec{n}) \right](x) \ldots \right\rangle = const \left\langle \mathcal{N}\left[P(\vec{n}, \partial\vec{n}) \right](x) \ldots \right\rangle \tag{106}$$

where $P(\vec{n}, \partial\vec{n})$ is arbitrary local polynomial of the fundamental fields and their derivatives, and $\mathcal{N}[\ldots]$ indicates renormalized normal product of the corresponding composite fields.

The first higher quantum conserved current (for $s = 3$ in the notations of (94), (95)) takes the following form:

$$\tilde{T}^{(4)} = \mathcal{N}\left[\left(\partial_-^2 \vec{n} \right)^2 \right] + a_1 \mathcal{N}\left[\left((\partial_- \vec{n})^2 \right)^2 \right] \tag{107}$$

$$\tilde{\Theta}^{(2)} = \left(\frac{1}{2} + a_2 \right) \mathcal{N}\left[(\partial_- \vec{n})^2 \, \sigma \right] + a_3 \partial_-^2 \sigma \tag{108}$$

where all coefficients $a_{1,2,3} = O(1/N)$ are expressed in terms of one-particle irreducible correlation functions and their derivatives in momentum space at zero external momenta. Their explicit form can be found order by order in $1/N$ from the renormalized $1/N$-diagram technique described above [52].

Let us stress, that the higher quantum conserved currents (107) and those for $s = 5, 7, \ldots$ do not have analogues in the classical conformally invariant theory [56].

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[8] Bogoliubov-Parasiuk-Hepp-Zimmermann [55]. As shown in [54], the (supersymmetric) nonlinear sigma-models (101) and (102) are renormalizable within the $1/N$ expansion also in $D = 3$ space-time dimensions in spite of their naive nonrenormalizability w.r.t. the ordinary coupling constant perturbation theory.

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